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Asymptotic behaviour of the solutions of second order functional differential equations

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ABSTRACT

This paper presents integral criteria to determine the asymptotic behaviour of the solutions of second order nonlinear differential equations of the type $y''(x) + q(x)f(y(x)) = 0$, with $q(x) > 0$ and $f(y)$ odd and positive for $y > 0$, as x tends to $+\infty$. It also compares them with the results obtained by Chanturia (1975) in [11] for the same problem.

1. Introduction

In a recent series of papers (see [1–3]) the authors described several methods to calculate bounds for the solutions of second order linear differential equations of the type

$$(p(x)y'(x))' + q(x)y(x) = 0, \quad x > x_0, \tag{1}$$

where $p(x)$ and $q(x)$ are strictly positive and continuously differentiable functions in an interval $[x_0, T]$ and $y(x_0)$ and $y'(x_0)$ are known, and applied them to determine the asymptotic behaviour of the solutions of (1) as $q(x)$ tends to $+\infty$ with x (see [3, Theorems 9 and 10]).

The theory to analyse the asymptotic behaviour of the solutions of these types of equations was first introduced by Armellini [4], Tonelli [5] and Sansone [6] for the case $p(x) = 1$ and $q(x)$ non-decreasing, that is, for the equation

$$y'' + q(x)y = 0, \quad x > x_0, \tag{2}$$

with $q(x)$ positive and non-decreasing. Armellini, Tonelli and Sansone extended the previous results of Milloux to determine the conditions for $q(x)$ under which all solutions of (2) vanish at $+\infty$. Further enhancements were provided by Opial [7,8], who analysed the case $q(x)$ as being non-monotonic, Kurzweil [9] and Hartman [10], who weakened the conditions for $q(x)$ originally stated by Armellini et al, and Teimuraz Chanturia [11].

Chanturia's result became a landmark in the literature of these types of problems since it addressed (and solved) the problem for the wider second order functional differential equation,

$$y'' + q(x)f(y) = 0, \quad x > x_0, \tag{3}$$

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with $q(x)$ being positive but non-necessarily non-decreasing and $f(y)$ being an odd non-decreasing function of y, which included (1) as a subcase. Chanturia showed sufficient conditions on $q(x)$ for $y(x)$ to tend to 0 or ∞ as x tends to $+\infty$, depending on whether $\lim_{x\to\infty} q(x) = +\infty$ or $\lim_{x\to\infty} q(x) = 0$, respectively.

The point in common for both Chanturia's and all the previous results is the use of the concept of regular growth, which can be broadly speaking translated as the non-existence of a sequence of intervals of a *small* measure where the growth of $q(x)$ is concentrated.

After Chanturia's work, further research on this topic has progressed in several directions, like refining the concept of regular growth of $q(x)$ in the linear case [12,13], increasing the complexity of the linear case by introducing small perturbations and advanced arguments [14,15] or step functions [16], or using the relationship of this problem with that of the asymptotics of the damped nonlinear oscillator, for which the application of the variational calculus has recently been quite fruitful [17–19]. However, in the opinion of the authors Chanturia's result still stands as the biggest step (as well as the widest approach) in the determination of the asymptotic behaviour of the solutions of (3).

Equations of the form (3) are quite common in physics. Examples include the equation of the oscillating pendulum with varying length or the Emden–Fowler equation, which appears in astrophysics and in the calculation of the distribution of electrons in the heavy atom. Furthermore, if $q(x) > 0$, by means of the change of variables

$$t = \phi(x) = \int_0^x \sqrt{q(s)}\mathrm{d}s,$$

it is possible to transform Eq. (3) in the equation

$$\ddot{z} + \frac{q'(\phi^{-1}(t))}{2q^{\frac{3}{2}}(\phi^{-1}(t))}\dot{z} + f(z) = 0, \quad t > \phi(x_0), \tag{4}$$

where $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}x$, $\dot{(\cdot)} = \mathrm{d}(\cdot)/\mathrm{d}t$ and ϕ^{-1} denotes the inverse function of ϕ. (4) includes cases like the Lane–Emden equation for composite stellar densities, the Haraux–Weissler equation or the equation of the damped nonlinear oscillator, all of which are associated to physical problems where the determination of sufficient conditions under which the solutions tend to 0 or $\pm\infty$ asymptotically becomes relevant.

The purpose of this paper will be to obtain results analogous to those of Chanturia in [11] but using integral criteria on $q(x)$ instead of the concept of regular growth. In order to facilitate the comparison, we will reuse the nomenclature proposed by Chanturia. Therefore, let $q(x)$ be a piecewise continuously differentiable function with a bounded variation in any closed interval of $[x_0, +\infty[$. We will define the operators

$$q_0^+(x) = q(x_0) + \left(\frac{\int_{x_0}^x |\mathrm{d}q(s)| + \int_{x_0}^x \mathrm{d}q(s)}{2}\right), \tag{5}$$

and

$$q_0^-(x) = \left(\frac{\int_{x_0}^x |\mathrm{d}q(s)| - \int_{x_0}^x \mathrm{d}q(s)}{2}\right). \tag{6}$$

It is evident that both $q_0^+(x)$ and $q_0^-(x)$ are non-decreasing functions in $[x_0, +\infty[$ and that

$$q(x) = q_0^+(x) - q_0^-(x). \tag{7}$$

Likewise, if $q(x)$ is a piecewise continuously differentiable function with a bounded variation in the interval $[x_0, +\infty[$ we will define the operators

$$q_\infty^+(x) = \left(\frac{\int_x^\infty |\mathrm{d}q(s)| + q(x)}{2}\right), \qquad q_\infty^-(x) = \left(\frac{\int_x^\infty |\mathrm{d}q(s)| - q(x)}{2}\right). \tag{8}$$

It is also evident that both $q_\infty^+(x)$ and $q_\infty^-(x)$ are non-increasing functions in $[x_0, \infty[$ and that

$$q(x) = q_\infty^+(x) - q_\infty^-(x). \tag{9}$$

Like Chanturia, we will use the functions q_0^+ and q_0^- in the proofs related to the case $\lim_{x\to+\infty} q(x) = +\infty$ and the functions q_∞^+ and q_∞^- in the proofs related to the case $\lim_{x\to+\infty} q(x) = 0$, respectively.

For the sake of clarity in presentation, let us recall Chanturia's main results as they were originally stated in [11]:

Theorem 1 (*Chanturia's Theorem 1*)**.** *Suppose that $q(x)$ is positive and piecewise continuously differentiable on $[x_0, +\infty[$ and satisfies*

$$\lim_{x\to+\infty} q(x) = +\infty; \qquad \int_{x_0}^\infty \frac{(q_0^-(x))'\mathrm{d}x}{q(x)} < \infty.$$

Suppose also that there exists an $\epsilon > 0$ *such that for every sequence* $\{\tau_i\}$ *which verifies*

$$0 \leq \tau_i \leq \tau_{i+1},\ i = 1, 2, \ldots; \quad \lim_{i\to+\infty} \tau_i = +\infty, \tag{10}$$

$$\lim_{k\to+\infty} \inf \sqrt{q_0^+(\tau_{2k})}(\tau_{2k} - \tau_{2k-1}) > 0, \qquad \lim_{k\to+\infty} \sup \sqrt{q_0^+(\tau_{2k-1})}(\tau_{2k} - \tau_{2k-1}) < \epsilon, \tag{11}$$

$$0 < \lim_{k\to+\infty} \inf \int_{\tau_{2k}}^{\tau_{2k+1}} \sqrt{q_0^+(x)}\mathrm{d}x \leq \lim_{k\to+\infty} \sup \int_{\tau_{2k}}^{\tau_{2k+1}} \sqrt{q_0^+(x)}\mathrm{d}x < +\infty, \tag{12}$$

one has

$$\sum_{n=1}^{\infty} (\ln q_0^+(\tau_{2k+1}) - \ln q_0^+(\tau_{2k})) = \infty. \tag{13}$$

Then all solutions of (3) *satisfy* $\lim_{x\to+\infty} y(x) = 0$.

Theorem 2 (*Chanturia's Theorem 5*). *Suppose that* $q(x)$ *is positive and piecewise continuously differentiable on* $[x_0, +\infty[$ *and satisfies*

$$\lim_{x\to+\infty} q(x) = 0; \qquad -\int_{x_0}^{\infty} \frac{(q_\infty^-(x))'\mathrm{d}x}{q(x)} < \infty.$$

Suppose also that there exists an $\epsilon > 0$ *such that for every sequence* $\{\tau_i\}$ *which verifies*

$$0 \leq \tau_i \leq \tau_{i+1},\ i = 1, 2, \ldots; \quad \lim_{i\to+\infty} \tau_i = +\infty, \tag{14}$$

$$\lim_{k\to+\infty} \inf \sqrt{q_\infty^+(\tau_{2k-1})}(\tau_{2k} - \tau_{2k-1}) > 0, \qquad \lim_{k\to+\infty} \sup \sqrt{q_\infty^+(\tau_{2k})}(\tau_{2k} - \tau_{2k-1}) < \epsilon, \tag{15}$$

$$0 < \lim_{k\to+\infty} \inf \int_{\tau_{2k}}^{\tau_{2k+1}} \sqrt{q_\infty^+(x)}\mathrm{d}x \leq \lim_{k\to+\infty} \sup \int_{\tau_{2k}}^{\tau_{2k+1}} \sqrt{q_\infty^+(x)}\mathrm{d}x < +\infty, \tag{16}$$

one has

$$\sum_{n=1}^{\infty} (\ln q_\infty^+(\tau_{2k}) - \ln q_\infty^+(\tau_{2k+1})) = \infty. \tag{17}$$

Then all solutions of (3) *satisfy* $\limsup_{x\to+\infty} y(x) = +\infty$ *and* $\liminf_{x\to+\infty} y(x) = -\infty$.

The organization of this paper is as follows. Section 2 will introduce sufficient conditions for a solution $y(x)$ of (3) to tend asymptotically to 0 in the case where $\lim_{x\to+\infty} q(x) = +\infty$. Likewise, Section 3 will introduce sufficient conditions for a solution $y(x)$ of (3) to have maxima and minima tending asymptotically to ∞ in the case where $\lim_{x\to+\infty} q(x) = 0$.

2. Asymptotic behaviour of $y(x)$ when $\lim_{x\to+\infty} q(x) = +\infty$

By Rolle's theorem and (3) it is clear that the zeroes of $y(x)$ and those of $y'(x)$ must be interlaced, i.e., between consecutive zeroes of $y(x)$ there must exist one (and only one) zero of $y'(x)$ and between consecutive zeroes of $y'(x)$ there must exist one zero of $y(x)$. Let us denote by $\{x_i\}$ the set of zeroes of $y(x)$ and $y'(x)$ such that $x_i > x_0$ and x_{2k} is a zero of $y(x)$.

The following theorem aims at giving conditions that bound (above and below) the distance between consecutive zeroes of $\{x_i\}$ as this sequence approaches ∞.

Theorem 3. *Let* $y(x)$ *be a solution of* (3) *and suppose that* $q(x)$ *is positive and piecewise continuously differentiable on* $]x_0, +\infty[$ *and satisfies*

$$\lim_{x\to+\infty} q(x) = +\infty; \qquad \int_{x_0}^{\infty} \frac{(q_0^-(x))'\mathrm{d}x}{q(x)} < \infty, \tag{18}$$

where $q_0^+(x)$ *and* $q_0^-(x)$ *are defined by* (5) *and* (6), *respectively. Suppose also that* $f(y)$ *is an odd function such that* $f(y) > 0$ *for* $y > 0$. *Let* $\{x_i\}$ *be the interlaced sequence of zeroes of* $y(x)$ *and* $y'(x)$. *Let* $\rho(x)$ *be the functional defined by*

$$\rho(y, x) = F(y(x)) + \frac{y'^2(x)}{2q(x)}, \tag{19}$$

where $F(y)=\int_0^y f(s)\mathrm{d}s$, and let us assume that

$$\lim_{x\to+\infty}\rho(x)=\rho_0>0. \tag{20}$$

Then the sequence $\{x_i\}$ verifies

$$x_i<x_{i+1},\ i=1,2,\ldots,\quad \lim_{i\to\infty}x_i=+\infty; \tag{21}$$

$$\lim_{i\to\infty}\sup\sqrt{q_0^+(x_i)}(x_{i+1}-x_i)<+\infty, \tag{22}$$

and

$$\lim_{i\to\infty}\inf\int_{x_i}^{x_{i+1}}\sqrt{q_0^+(x)}\mathrm{d}x>0. \tag{23}$$

Proof. Following an argument similar to that used by Chanturia in Theorem 1 (see [11]), we will first prove that $\lim_{x\to+\infty}\frac{q(x)}{q_0^+(x)}=1$, i.e., $\lim_{x\to+\infty}\frac{q_0^-(x)}{q_0^+(x)}=0$ (note that from (7) one has that $q(x)=q_0^+(x)-q_0^-(x)$). Thus, let $z_1,x\in]x_0,+\infty[$ be such that $x>z_1$. One has that

$$\frac{q_0^-(x)}{q_0^+(x)}=\frac{q_0^-(z_1)}{q_0^+(x)}+\frac{\int_{z_1}^x(q_0^-(s))'\mathrm{d}s}{q_0^+(x)}\le\frac{q_0^-(z_1)}{q_0^+(x)}+\int_{z_1}^x\frac{(q_0^-(s))'}{q_0^+(s)}\mathrm{d}s,\quad x\in]z_1,+\infty[, \tag{24}$$

since, from (5), $q_0^+(x)$ is increasing in $]z_1,+\infty[$. Applying hypothesis (18) to (24) one gets

$$\frac{q_0^-(x)}{q_0^+(x)}\le\frac{q_0^-(z_1)}{q_0^+(x)}+\int_{z_1}^{+\infty}\frac{(q_0^-(s))'}{q_0^+(s)}\mathrm{d}s<\infty,\quad x\in]z_1,+\infty[. \tag{25}$$

Since the integral $\int_{z_1}^{+\infty}\frac{(q_0^-(s))'}{q_0^+(s)}\mathrm{d}s$ is convergent for any $z_1\in]x_0,+\infty[$, for every $\epsilon>0$ we can always pick z_1 such that

$$\int_{z_1}^{+\infty}\frac{(q_0^-(s))'}{q_0^+(s)}\mathrm{d}s<\epsilon. \tag{26}$$

From (25) and (26) one gets

$$\frac{q_0^-(x)}{q_0^+(x)}\le\frac{q_0^-(z_1)}{q_0^+(x)}+\epsilon,\quad x\in]z_1,+\infty[. \tag{27}$$

Given that $\lim_{x\to+\infty}q_0^+(x)=\lim_{x\to+\infty}q(x)=+\infty$, this makes

$$\lim_{x\to+\infty}\frac{q_0^-(x)}{q_0^+(x)}\le\epsilon, \tag{28}$$

for every $\epsilon>0$, that is

$$\lim_{x\to+\infty}\frac{q_0^-(x)}{q_0^+(x)}=0. \tag{29}$$

From (29) one yields $\lim_{x\to+\infty}\frac{q(x)}{q_0^+(x)}=1$. Then, for every $\gamma>1$ we can always find an x^* such that

$$\frac{q_0^+(x)}{\gamma^2}\le q(x)\le q_0^+(x),\quad x>x^*. \tag{30}$$

Next we will prove (22) and (23). To this end let us define $\{x_i,i\ge1\}$ as the subsequence of zeroes of $y(x)$ and $y'(x)$ such that x_1 is a zero of $y(x)$ and $x_i>x^*$ for every $i\ge1$. It is clear that $\{x_i,i\ge1\}$ satisfies (21).

Now, from (20), for any $\delta>0$ there must exist an $x^{**}>x^*$ such that

$$\rho_0-\delta<\rho(x)<\rho_0+\delta,\quad x>x^{**}. \tag{31}$$

From (19) and (31) we can define

$$\begin{aligned}Y_m&=\min\{|y(x_{2k+1})|,\ x_{2k+1}>x^{**}\},\\ Y_M&=\max\{|y(x_{2k+1})|,\ x_{2k+1}>x^{**}\},\end{aligned} \tag{32}$$

so that $Y_m > 0$ and

$$\frac{y'^2(x)}{2q(x)} < \rho_0 + \delta \Rightarrow \sqrt{q(x)} > \sqrt{\frac{y'^2(x)}{2(\rho_0 + \delta)}}, \quad x > x^{**},$$

and, as a consequence

$$\int_{x_i}^{x_{i+1}} \sqrt{q(x)}\mathrm{d}x > \frac{\int_{x_i}^{x_{i+1}} |y'(x)|\mathrm{d}x}{\sqrt{2(\rho_0 + \delta)}} = \frac{\max(|y(x_i)|, |y(x_{i+1})|)}{\sqrt{2(\rho_0 + \delta)}} > \frac{Y_m}{\sqrt{2(\rho_0 + \delta)}} > 0, \quad i \geq 1. \tag{33}$$

This proves (23). In order to prove (22), let us define the sequence $\{\tau_i\}$ already introduced by Chanturia such that

$$\tau_{i+1} > \tau_i, \; i = 1, 2, \dots, \quad \lim_{i\to\infty} \tau_i = +\infty, \quad \tau_1 > x^{**}, \tag{34}$$

and

$$F(\tau_i) = F(\tau_{i+1}) = \rho_0 - 2\delta, \quad i \geq 1. \tag{35}$$

Since x_{2k} is a zero of $y(x)$, x_{2k+1} must be either a maximum or a minimum of $y(x)$. We will assume without loss of generality that

$$\tau_{2k} < x_{2k+1} < \tau_{2k+1}. \tag{36}$$

Then it is clear from (19), (31) and (35) that

$$\frac{y'^2(x)}{2q(x)} > \delta, \quad x \in]\tau_{2k-1}, \tau_{2k}[.$$

Therefore

$$\begin{aligned}\sqrt{q_0^+(\tau_{2k})}(\tau_{2k} - x_{2k}) &< \gamma \int_{x_{2k}}^{\tau_{2k}} \sqrt{q(x)}\mathrm{d}x < \gamma \int_{x_{2k}}^{\tau_{2k}} \frac{|y'(x)|}{\sqrt{2\delta}}\mathrm{d}x \\ &< \frac{\gamma\,|y(x_{2k+1})|}{\sqrt{2\delta}} \leq \frac{\gamma Y_M}{\sqrt{2\delta}}.\end{aligned} \tag{37}$$

On the other hand, from (3) and (30) one has

$$(y'(x))^2 = \int_x^{x_{2k+1}} q(x)f(y(x))y'(x)\mathrm{d}x > \frac{q_0^+(x_{2k})}{\gamma^2}(F(y(x_{2k+1})) - F(y(x))).$$

In consequence, if we define

$$c = \min\{|f(y)|, \; \rho_0 - 2\delta < F(y) < \rho_0 + \delta\},$$

one gets

$$\begin{aligned}\sqrt{q_0^+(x_{2k})}(x_{2k+1} - \tau_{2k}) &< \gamma \int_{\tau_{2k}}^{x_{2k+1}} \frac{|y'(x)|\mathrm{d}x}{\sqrt{F(y(x_{2k+1})) - F(y(x))}} \\ &< \frac{\gamma}{c} \int_{\tau_{2k}}^{x_{2k+1}} \frac{|f(y)y'(x)|\mathrm{d}x}{\sqrt{F(y(x_{2k+1})) - F(y(x))}} \\ &= \frac{2\gamma}{c}\sqrt{F(y(x_{2k+1})) - F(y(\tau_{2k}))} \leq \frac{2\gamma\sqrt{3\delta}}{c}.\end{aligned} \tag{38}$$

If we add (37) and (38) one yields

$$\sqrt{q_0^+(x_{2k})}(x_{2k+1} - x_{2k}) \leq \frac{\gamma\,|y(x_{2k+1})|}{\sqrt{2\delta}} \leq \frac{\gamma Y_M}{\sqrt{2\delta}} + \frac{2\gamma\sqrt{3\delta}}{c}. \tag{39}$$

A similar argument leads to

$$\sqrt{q_0^+(x_{2k-1})}(x_{2k} - x_{2k-1}) \leq \frac{\gamma\,|y(x_{2k+1})|}{\sqrt{2\delta}} \leq \frac{\gamma Y_M}{\sqrt{2\delta}} + \frac{2\gamma\sqrt{3\delta}}{c}. \tag{40}$$

From (39) and (40) one gets (22). □

Theorem 4. *Suppose that $q(x)$ is positive and piecewise continuously differentiable on $[x_0, +\infty[$ such that $\lim_{x\to+\infty} q(x) = +\infty$. Let the functions $q_0^+(x)$ and $q_0^-(x)$ be defined as in (5) and (6). Suppose also that $f(y)$ is an odd function such that $f(y) > 0$ for $y > 0$. If*

$$\int_{x_0}^{+\infty} \frac{(q_0^-(x))'}{q(x)} \mathrm{d}x < \infty \tag{41}$$

and for each sequence $\{x_i\}$ that satisfies (21)–(23), there exists a positive piecewise continuous function $h(x)$ such that

$$\sum_{i=1}^{\infty} \frac{I_i}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{h(x)} \mathrm{d}x} = \infty, \tag{42}$$

where

$$I_{2k+1} = \frac{\int_{x_{2k+1}}^{x_{2k+2}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x \le s \le x_{2k+2}\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2k+1}}^{x_{2k+2}} \frac{h(x)}{q^2(x)} \mathrm{d}x}, \quad k \ge 0, \tag{43}$$

$$I_{2k} = \frac{\int_{x_{2k}}^{x_{2k+1}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x_{2k} \le s \le x\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2k}}^{x_{2k+1}} \frac{h(x)}{q^2(x)} \mathrm{d}x}, \quad k \ge 1, \tag{44}$$

then all solutions of (3) satisfy $\lim_{x\to+\infty} y(x) = 0$.

Proof. The proof follows a similar argument to that of [3, Theorem 10]. Thus, let the functional $\rho(x)$ be defined by (19). Differentiating $\rho(x)$ one has, from (3), that

$$\rho'(x) = -\frac{q'(x)}{2q^2(x)} y'^2(x) = -\frac{q'(x)}{2q^2(x)} \frac{y'^2(x)}{\rho(x)} \rho(x). \tag{45}$$

Dividing both sides of (45) by $\rho(x)$, integrating from x_0 to x and using (5) and (6) one gets

$$\begin{aligned} \rho(x) &= \rho(x_0) \exp\left(-\int_{x_0}^{x} \frac{q'(x)}{2q^2(x)} \frac{y'^2(x)}{\rho(x)} \mathrm{d}x\right) \\ &= \rho(x_0) \exp\left(\int_{x_0}^{x} \frac{(q_0^-(x))'}{q(x)} \frac{y'^2(x)}{2q(x)\rho(x)} \mathrm{d}x\right) \exp\left(-\int_{x_0}^{x} \frac{(q_0^+(x))'}{2q^2(x)} \frac{y'^2(x)}{\rho(x)} \mathrm{d}x\right). \end{aligned} \tag{46}$$

Given that $\frac{y'^2(x)}{2q(x)} \le \rho(x)$ from (19) and $\int_{x_0}^{+\infty} \frac{(q_0^-(x))'}{q(x)} \mathrm{d}x = G < +\infty$ from hypothesis (41), from (46) one has that $\rho(x)$ is a product of a monotonic increasing function which has $\exp G$ as a limit as $x \to +\infty$ and a monotonic decreasing function which is always positive. Therefore $\rho(x)$ must have a limit $\rho_0 \ge 0$ as $x \to +\infty$,

$$\rho(x) \le \rho(x_0) \exp G, \quad x > x_0, \tag{47}$$

and

$$\rho_0 \le \rho(x_0) \exp G. \tag{48}$$

Let us now focus on obtaining an upper bound for the second exponential of (46).
From (47) it is straightforward to show that

$$\int_{x_0}^{\infty} \frac{(q_0^+(x))'}{2q^2(x)} \frac{y'^2(x)}{\rho(x)} \mathrm{d}x \ge \frac{1}{2\rho(x_0)\exp G} \int_{x_0}^{\infty} \frac{(q_0^+(x))'}{q^2(x)} y'^2(x) \mathrm{d}x. \tag{49}$$

Taking into account the sequence $\{x_i\}$ of zeroes of $y(x)$ and $y'(x)$ defined before, for any positive and piecewise continuous function $h(x)$ the mean value theorem for integrals establishes that there must be a sequence $\{\xi_i,\ i \ge 1\}$ such that $\xi_i \in]x_i, x_{i+1}[$ and

$$\begin{aligned} \int_{x_0}^{\infty} \frac{(q_0^+(x))'}{2q^2(x)} \frac{y'^2(x)}{\rho(x)} \mathrm{d}x &\ge \int_{x_0}^{x_1} \frac{(q_0^+(x))'}{q^2(x)} \frac{y'^2(x)}{2\rho(x)} \mathrm{d}x + \frac{1}{2\rho(x_0)\exp G} \sum_{i=1}^{\infty} \frac{(q_0^+(\xi_i))'}{h(\xi_i)} \int_{x_i}^{x_{i+1}} \frac{h(x)}{q^2(x)} y'^2(x) \mathrm{d}x \\ &\ge K + \frac{1}{2\rho(x_0)\exp G} \sum_{i=1}^{\infty} \frac{(q_0^+(\xi_i))'}{h(\xi_i)} \int_{x_i}^{x_{i+1}} \frac{h(x)}{q^2(x)} y'^2(x) \mathrm{d}x, \end{aligned} \tag{50}$$

where K is a positive constant and $\frac{(q_0^+(\xi_i))'}{h(\xi_i)}$ can be bounded below, using [2, Theorems 3 and 4], by

$$\frac{(q_0^+(\xi_{2k}))'}{h(\xi_{2k})} \geq \frac{\int_{x_{2k}}^{x_{2k+1}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x_{2k} \leq s \leq x\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2k}}^{x_{2k+1}} \frac{h(x)}{q^2(x)} \mathrm{d}x}, \tag{51}$$

$$\frac{(q_0^+(\xi_{2k+1}))'}{h(\xi_{2k+1})} \geq \frac{\int_{x_{2k+1}}^{x_{2k+2}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x \leq s \leq x_{2k+2}\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2k+1}}^{x_{2k+2}} \frac{h(x)}{q^2(x)} \mathrm{d}x}, \tag{52}$$

since x_{2k} are zeroes of $y(x)$ and x_{2k+1} zeroes of $y'(x)$.

If we apply Yang's version of Opial's inequality [20, Theorems 3 and 3'] to (50) we get

$$\int_{x_0}^{+\infty} \frac{(q_0^+(x))'}{2q^2(x)} \frac{y'^2(x)}{\rho(x)} \mathrm{d}x \geq K + \frac{1}{2\rho(x_0)\exp G} \sum_{i=1}^{\infty} \frac{(q_0^+(\xi_i))'}{h(\xi_i)} \frac{y_i^2}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{h(x)} \mathrm{d}x}, \tag{53}$$

where $y_{2k}^2 = y_{2k+1}^2 = y^2(x_{2k+1})$. We will show by contradiction that $\lim_{x\to+\infty} \rho(x) = \rho_0 \neq 0$ is incompatible with the hypothesis (42) of this theorem.

Thus, let us assume that $\rho_0 > 0$. Then, from (19), one has that $\lim_{j\to+\infty} y_j^2 = (F^{-1}(\rho_0))^2 > 0$, because $F(y)$ is a strictly increasing function. As a consequence there must exist a minimum $Y > 0$ for the sequence $\{y_j^2\}$. In that case from (48) and (53) one gets

$$\rho_0 \leq \rho(x_0)\exp(G-K)\exp\left(-\frac{Y}{2\rho(x_0)\exp G} \sum_{i=1}^{\infty} \frac{(q_0^+(\xi_i))'}{h(\xi_i)} \frac{1}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{h(x)} \mathrm{d}x}\right). \tag{54}$$

Since from Theorem 3 the sequence of zeroes $\{x_i\}$ satisfies (21)–(23), from hypotheses (42) and (54) one gets that $\rho_0 = 0$ in contradiction with our assumption. This proves the theorem. □

Based on this theorem, the following corollaries provide more practical criteria to determine the asymptotic behaviour of the solutions of (3).

Corollary 1. *Under the same hypotheses of Theorem 4, if for each sequence $\{x_i,\ i \geq 1\}$ which satisfies (21)–(23), there exists a positive and piecewise continuous function $h(x)$ such that either*

$$\sum_{i=1}^{\infty} \frac{\int_{x_{2i}}^{x_{2i+1}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x_{2i} \leq s \leq x\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2i}}^{x_{2i+1}} \frac{h(x)}{q^2(x)} \mathrm{d}x \int_{x_{2i}}^{x_{2i+1}} \frac{q^2(x)}{h(x)} \mathrm{d}x} = \infty, \tag{55}$$

or

$$\sum_{i=1}^{\infty} \frac{\int_{x_{2i+1}}^{x_{2i+2}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x \leq s \leq x_{2i+2}\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2i+1}}^{x_{2i+2}} \frac{h(x)}{q^2(x)} \mathrm{d}x \int_{x_{2i+1}}^{x_{2i+2}} \frac{q^2(x)}{h(x)} \mathrm{d}x} = \infty, \tag{56}$$

then all solutions of (3) satisfy $\lim_{x\to+\infty} y(x) = 0$.

Proof. The proof is immediate since the series (42) of Theorem 4 is the sum of the series (55) and (56) for any sequence $\{x_i\}$ satisfying (21)–(23). Therefore it is clear that if either (55) or (56) are infinite then (42) must also be infinite and vice versa. □

Corollary 2. *Under the same hypotheses of Theorem 4, if for each sequence $\{x_i,\ i \geq 1\}$ which satisfies (21)–(23), there exists a positive and piecewise continuous function $h(x)$ such that*

$$\sum_{i=1}^{\infty} \min\left\{\frac{(q_0^+(x))'}{h(x)},\ x \in [x_i, x_{i+1}]\right\} \frac{1}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{h(x)} \mathrm{d}x} = \infty, \tag{57}$$

then all solutions of (3) satisfy $\lim_{x\to+\infty} y(x) = 0$.

Proof. The proof is also straightforward from (42)–(44) and taking into account that

$$\min\left\{\frac{(q_0^+(x))'}{h(x)},\ x \in [x_{2k}, x_{2k+1}]\right\} \leq \frac{\int_{x_{2k}}^{x_{2k+1}} \min\left(\frac{(q_0^+(s))'}{h(s)},\ x_{2k} \leq s \leq x\right) \frac{h(x)}{q^2(x)} \mathrm{d}x}{\int_{x_{2k}}^{x_{2k+1}} \frac{h(x)}{q^2(x)} \mathrm{d}x}, \tag{58}$$

and

$$\min\left\{\frac{(q_0^+(x))'}{h(x)},\ x\in[x_{2k-1},x_{2k}]\right\}\le\frac{\int_{x_{2k-1}}^{x_{2k}}\min\left(\frac{(q_0^+(s))'}{h(s)},\ x\le s\le x_{2k}\right)\frac{h(x)}{q^2(x)}\mathrm{d}x}{\int_{x_{2k-1}}^{x_{2k}}\frac{h(x)}{q^2(x)}\mathrm{d}x}.\quad\square \tag{59}$$

Corollary 3. *Under the same hypotheses of Theorem 4, if for all sequences $\{x_i,\ i\ge 1\}$ which satisfy (21)–(23), one has that*

$$\sum_{i=1}^{\infty}\frac{1}{\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{(q_0^+(x))'}\mathrm{d}x}=\infty, \tag{60}$$

where the terms $\frac{1}{\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{(q_0^+(x))'}\mathrm{d}x}$ are substituted by zero if $(q_0^+(x))'$ (ergo $q'(x)$) vanishes in $]x_i,x_{i+1}[$, then all solutions of (3) satisfy $\lim_{x\to+\infty}y(x)=0$.

Proof. The proof is immediate from Corollary 2 by taking $h(x)=1$ in those intervals $[x_i,x_{i+1}]$ having at least one point where $(q_0^+(x))'$ vanishes and $h(x)=(q_0^+(x))'$ in the rest. $\square$

Remark 1. As can be seen from Theorems 1–4 and the corollaries stated afterwards, the principal advantage of the results of the present paper versus those of Chanturia are the simplification of the criteria to choose the fundamental sequences, $\{x_i\}$ in our case and $\{\tau_i\}$ in the case of Chanturia. In the latter case the distance between τ_{2k} and τ_{2k+1} is not related to the distance between τ_{2k+1} and τ_{2k+2}, whereas here the criteria for the distance between x_i and x_{i+1} follows a similar pattern in terms of i.

Example 1. Let us consider the case $q(x)=x^\alpha$, where $\alpha>0$. Since $q(x)$ is monotonic increasing it is clear that $q_0^+(x)=q(x)$. In order to apply Corollary 3 we need to calculate $\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{q'(x)}\mathrm{d}x$. Thus:

$$\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{q'(x)}\mathrm{d}x=\int_{x_i}^{x_{i+1}}\frac{x^{2\alpha}}{\alpha x^{\alpha-1}}\mathrm{d}x=\frac{1}{\alpha(\alpha+2)}[x_{i+1}^{\alpha+2}-x_i^{\alpha+2}].$$

Then

$$\sum_{i=1}^{\infty}\frac{1}{\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{q'(x)}\mathrm{d}x}=\sum_{i=1}^{\infty}\frac{\alpha(\alpha+2)}{x_{i+1}^{\alpha+2}-x_i^{\alpha+2}}. \tag{61}$$

We will show that the non-convergence of (61) is related to the non-convergence of the series

$$\sum_{i=1}^{\infty}\frac{1}{x_i^{\alpha+1}(x_{i+1}-x_i)}. \tag{62}$$

Indeed, let us recall Newton's binomial theorem ([21, pp. 68 and 242]), which for any real number r and complex numbers x and y such that $|x|>|y|$ states that

$$(x+y)^r=\sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}y^k.$$

If we apply this theorem to the denominator of (61) then one gets

$$\begin{aligned}x_{i+1}^{\alpha+2}-x_i^{\alpha+2}&=\sum_{k=0}^{\infty}\binom{\alpha+2}{k}x_i^{\alpha+2-k}(x_{i+1}-x_i)^k-x_i^{\alpha+2}\\&=\sum_{k=1}^{\infty}\binom{\alpha+2}{k}x_i^{\alpha+2-k}(x_{i+1}-x_i)^k\\&=(\alpha+2)x_i^{\alpha+1}(x_{i+1}-x_i)+\sum_{k=2}^{\infty}\binom{\alpha+2}{k}x_i^{\alpha+2-k}(x_{i+1}-x_i)^k\\&=(\alpha+2)x_i^{\alpha+1}(x_{i+1}-x_i)\left(1+\frac{\sum_{k=2}^{\infty}\binom{\alpha+2}{k}x_i^{1-k}(x_{i+1}-x_i)^{k-1}}{\alpha+2}\right).\end{aligned} \tag{63}$$

Since the sequence $\{x_i\}$ tends to $+\infty$ and the difference $x_{i+1} - x_i$ tends to 0, the second term in the parenthesis of (63) must tend to 0 as i approaches ∞, i.e., for any $\epsilon > 0$ there must exist an $i^* > 1$ such that for $i > i^*$, the mentioned second term is upper bounded by ϵ. This implies that there must exist a maximum $S > 0$ for those second terms such that

$$\frac{\sum_{k=2}^{\infty}\binom{\alpha+2}{k} x_i^{1-k}(x_{i+1}-x_i)^{k-1}}{\alpha+2} < S, \quad i = 1, 2, \ldots, \infty.$$

Therefore

$$\sum_{i=1}^{\infty} \frac{\alpha(\alpha+2)}{(\alpha+2)x_i^{\alpha+1}(x_{i+1}-x_i)\left(1+\frac{\sum_{k=2}^{\infty}\binom{\alpha+2}{k}x_i^{1-k}(x_{i+1}-x_i)^{k-1}}{\alpha+2}\right)} > \sum_{i=1}^{\infty}\frac{\alpha}{x_i^{\alpha+1}(x_{i+1}-x_i)(1+S)}. \tag{64}$$

From (64) it is evident that if (62) does not converge (61) will not do it either.

Let us now focus on determining the non-convergence of the series (62). Since the sequence $\{x_i\}$ must satisfy conditions (21)–(23), there must exist an $M > 0$ such that

$$x_i^{\alpha}(x_{i+1}-x_i)^2 \le M.$$

If we apply this to (62) then one yields

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{x_i^{\alpha+1}(x_{i+1}-x_i)} &\ge \frac{1}{M}\sum_{i=1}^{\infty}\frac{(x_{i+1}-x_i)}{x_i}\\ &\ge \frac{1}{M}\sum_{i=1}^{\infty}\int_{x_i}^{x_{i+1}}\frac{dx}{x} = \frac{1}{M}\int_{x_1}^{\infty}\frac{dx}{x} = \frac{1}{M}[\ln x]_{x_1}^{\infty} = +\infty.\end{aligned} \tag{65}$$

(65) implies that the series of Corollary 3 is always infinite if the sequence $\{x_i\}$ satisfies (21)–(23). Therefore, when $q(x) = x^{\alpha}$, $\alpha > 0$, all solutions of (3) tend to 0 as $x \to +\infty$.

3. Asymptotic behaviour of $y(x)$ when $\lim_{x\to+\infty} q(x) = 0$

As in the previous section, we will begin with a theorem describing the asymptotic behaviour of the distance between consecutive zeroes of the sequence $\{x_i\}$. Since the existence of this sequence is not guaranteed in the case $q(x) \to 0$, we will explicitly impose the condition of oscillation of the solution $y(x)$ of (3) on $]x_0, +\infty[$ for the theorems stated hereinafter to be applicable (conditions to determine the oscillation of $y(x)$ can be found for instance in [22]).

Theorem 5. *Suppose that (3) has a solution $y(x)$ oscillatory on $]x_0, +\infty[$. Let $q(x)$ be positive and piecewise continuously differentiable with a bounded variation on $]x_0, +\infty[$ such that*

$$\lim_{x\to+\infty} q(x) = 0; \qquad -\int_{x_0}^{\infty}\frac{(q_{\infty}^{-}(x))'dx}{q(x)} < \infty, \tag{66}$$

where $q_{\infty}^{+}(x)$ and $q_{\infty}^{-}(x)$ are defined by (8). Let $f(y)$ be an odd function such that $f(y) > 0$ for $y > 0$ and $\int_0^{+\infty} f(y)dy = +\infty$. Let $\{x_i\}$ be the interlaced sequence of zeroes of $y(x)$ and $y'(x)$. Let $\rho(x)$ be the functional defined by (19) and let us assume that

$$\lim_{x\to+\infty}\rho(x) = \rho_0 > 0. \tag{67}$$

Then the sequence $\{x_i\}$ verifies

$$x_i < x_{i+1}, \; i = 1, 2, \ldots, \quad \lim_{i\to\infty} x_i = +\infty; \tag{68}$$

$$\limsup_{i\to\infty}\sqrt{q_{\infty}^{+}(x_{i+1})}(x_{i+1}-x_i) < +\infty, \tag{69}$$

and

$$\liminf_{i\to\infty}\int_{x_i}^{x_{i+1}}\sqrt{q_{\infty}^{+}(x)}dx > 0. \tag{70}$$

Proof. The proof of this theorem mimics most of that of Theorem 3 just taking into account that, from (8), $q_{\infty}^{+}(x)$ and $q_{\infty}^{+}(x)$ are monotonic decreasing. The only part which is slightly different is the proof of $\lim_{x\to+\infty}\frac{q_{\infty}^{-}(x)}{q_{\infty}^{+}(x)} = 0$.

In order to prove it let $z_1, x \in]x_0, +\infty[$ be such that $z_1 > x$. One has that

$$\frac{q_\infty^-(x)}{q_\infty^+(x)} = \frac{q_\infty^-(z_1)}{q_\infty^+(x)} - \frac{\int_x^{z_1}(q_\infty^-(s))'\mathrm{d}s}{q_\infty^+(x)} \le \frac{q_\infty^-(z_1)}{q_\infty^+(x)} - \int_x^{z_1}\frac{(q_\infty^-(s))'}{q_\infty^+(s)}\mathrm{d}s, \quad x \in]z_1, +\infty[. \tag{71}$$

Applying hypothesis (66) to (71) and taking into account that $\lim_{z_1\to+\infty} q_\infty^-(z_1) = 0$ one gets

$$\frac{q_\infty^-(x)}{q_\infty^+(x)} \le -\int_x^{+\infty}\frac{(q_\infty^-(s))'}{q_\infty^+(s)}\mathrm{d}s < \infty. \tag{72}$$

Since the integral $-\int_x^{+\infty}\frac{(q_\infty^-(s))'}{q_\infty^+(s)}\mathrm{d}s$ is convergent for any $x \in]x_0, +\infty[$, for every $\epsilon > 0$ there is always an $x^* > x_0$ such that, if $x > x^*$ one yields

$$-\int_x^{+\infty}\frac{(q_\infty^-(s))'}{q_\infty^+(s)}\mathrm{d}s < \epsilon. \tag{73}$$

From (72) and (73) it is immediate to show that

$$\frac{q_\infty^-(x)}{q_\infty^+(x)} < \epsilon, \quad x \in]x^*, +\infty[. \tag{74}$$

This proves the theorem. □

Note that from (74), as in Theorem 3, for any $\gamma > 1$ there must exist x^* such that

$$\frac{q_\infty^+(x)}{\gamma} \le q(x) \le q_\infty^+(x), \quad x \in]x^*, +\infty[. \tag{75}$$

Theorem 6. *Let $q(x)$ be a positive and piecewise continuously differentiable function with a bounded variation on $[x_0, +\infty[$ such that $\lim_{x\to+\infty} q(x) = 0$. Let the functions $q_\infty^+(x)$ and $q_\infty^-(x)$ be defined as in (8). Suppose also that $f(y)$ is an odd function such that $f(y) > 0$ for $y > 0$ and $\int_0^{+\infty} f(y)\mathrm{d}y = +\infty$. If*

$$-\int_{x_0}^{+\infty}\frac{(q_\infty^-(x))'}{q(x)}\mathrm{d}x < +\infty \tag{76}$$

and for each sequence $\{x_i\}$ that satisfies (68)–(70) there exists a positive piecewise continuous function $h(x)$ such that

$$\sum_{i=1}^{\infty}\frac{I_i}{\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{h(x)}\mathrm{d}x} = +\infty, \tag{77}$$

where

$$I_{2k} = \frac{\int_{x_{2k}}^{x_{2k+1}}\min\left(\frac{-(q_\infty^+(s))'}{h(s)},\ x_{2k} \le s \le x\right)\frac{h(x)}{q^2(x)}\mathrm{d}x}{\int_{x_{2k}}^{x_{2k+1}}\frac{h(x)}{q^2(x)}\mathrm{d}x}, \quad k \ge 0, \tag{78}$$

$$I_{2k+1} = \frac{\int_{x_{2k+1}}^{x_{2k+2}}\min\left(\frac{-(q_\infty^+(s))'}{h(s)},\ x \le s \le x_{2k+2}\right)\frac{h(x)}{q^2(x)}\mathrm{d}x}{\int_{x_{2k+1}}^{x_{2k+2}}\frac{h(x)}{q^2(x)}\mathrm{d}x}, \quad k \ge 0, \tag{79}$$

then all solutions of (3) which are oscillatory on $]x_0, +\infty[$, if they exist, satisfy $\limsup_{x\to+\infty} y(x) = +\infty$ and $\liminf_{x\to+\infty} y(x) = -\infty$.

Proof. As in Theorem 4, from (9) it is easy to show that the functional $\rho(x)$ defined in (19) can be expressed as

$$\begin{aligned}\rho(x) &= \rho(x_0)\exp\left(-\int_{x_0}^{x}\frac{q'(x)}{2q^2(x)}\frac{y'^2(x)}{\rho(x)}\mathrm{d}x\right)\\ &= \rho(x_0)\exp\left(\int_{x_0}^{x}\frac{(q_\infty^-(x))'}{q(x)}\frac{y'^2(x)}{2q(x)\rho(x)}\mathrm{d}x\right)\exp\left(-\int_{x_0}^{x}\frac{(q_\infty^+(x))'}{2q^2(x)}\frac{y'^2(x)}{\rho(x)}\mathrm{d}x\right).\end{aligned} \tag{80}$$

Since $\frac{y'^2(x)}{2q(x)} \le \rho(x)$ from (19) and $\int_{x_0}^{+\infty}\frac{(q_\infty^-(x))'}{q(x)}\mathrm{d}x = G > -\infty$ from hypothesis (76), from (80) one has that $\rho(x)$ is a product of a monotonic decreasing function which has $\exp G$ as a limit as $x \to +\infty$ and a monotonic increasing function. Therefore $\rho(x)$ must have a limit $0 < \rho_0 \le \infty$ as $x \to +\infty$.

We will show by contradiction that $\lim_{x\to+\infty}\rho(x)=\rho_0<\infty$ is incompatible with the hypothesis (77) of this theorem for $y(x)$ oscillatory on $]x_0,+\infty[$. Thus, let us assume that $\rho_0<\infty$ and let us focus on obtaining a lower bound for the second exponential of (80).

If $y(x)$ is oscillatory on $]x_0,+\infty[$ there must exist a sequence $\{x_i\}$ of zeroes of $y(x)$, $y'(x)$. Furthermore, there must exist an even index $i_1\geq 1$ such that for any $\delta>0$ one has

$$\rho_0-\delta<\rho(x)<\rho_0+\delta,\quad x>x_{i_1}. \tag{81}$$

Then, from (80), (81) and the mean value theorem for integrals, for any positive and piecewise continuous function $h(x)$ we get

$$\begin{aligned}-\int_{x_0}^{\infty}\frac{(q_\infty^+(x))'}{2q^2(x)}\frac{y'^2(x)}{\rho(x)}dx &\geq -\int_{x_0}^{x_{i_1}}\frac{(q_\infty^+(x))'}{q^2(x)}\frac{y'^2(x)}{2\rho(x)}dx-\sum_{k=\frac{i_1}{2}}^{\infty}\int_{x_{2k}}^{x_{2k+1}}\frac{(q_\infty^+(x))'}{2q^2(x)}\frac{y'^2(x)}{(\rho_0+\delta)}dx\\ &\quad-\sum_{k=\frac{i_1}{2}}^{\infty}\int_{x_{2k+1}}^{x_{2k+2}}\frac{(q_\infty^+(x))'}{2q^2(x)}\frac{y'^2(x)}{(\rho_0+\delta)}dx\\ &\geq K_1+\sum_{k=\frac{i_1}{2}}^{\infty}\frac{-(q_\infty^+(\xi_{2k}))'}{h(\xi_{2k})}\frac{1}{2(\rho_0+\delta)}\int_{x_{2k}}^{x_{2k+1}}\frac{h(x)y'^2(x)}{q^2(x)}dx\\ &\quad+\sum_{k=\frac{i_1}{2}}^{\infty}\frac{-(q_\infty^+(\xi_{2k+1}))'}{h(\xi_{2k+1})}\frac{1}{2(\rho_0+\delta)}\int_{x_{2k+1}}^{x_{2k+2}}\frac{h(x)y'^2(x)}{q^2(x)}dx,\end{aligned} \tag{82}$$

where K_1 is a positive constant and $-\frac{(q_\infty^+(\xi_i))'}{h(\xi_i)}$ can be bounded below, using [2, Theorems 3 and 4], by

$$\frac{-(q_\infty^+(\xi_{2k}))'}{h(\xi_{2k})}\geq\frac{\int_{x_{2k}}^{x_{2k+1}}\min\left(\frac{-(q_\infty^+(s))'}{h(s)},\ x_{2k}\leq s\leq x\right)\frac{h(x)}{q^2(x)}dx}{\int_{x_{2k}}^{x_{2k+1}}\frac{h(x)}{q^2(x)}dx}, \tag{83}$$

$$\frac{-(q_\infty^+(\xi_{2k+1}))'}{h(\xi_{2k+1})}\geq\frac{\int_{x_{2k+1}}^{x_{2k+2}}\min\left(\frac{-(q_\infty^+(s))'}{h(s)},\ x\leq s\leq x_{2k+2}\right)\frac{h(x)}{q^2(x)}dx}{\int_{x_{2k+1}}^{x_{2k+2}}\frac{h(x)}{q^2(x)}dx}, \tag{84}$$

since x_{2k} are zeroes of $y(x)$ and x_{2k+1} zeroes of $y'(x)$.

If we apply Yang's version of Opial's inequality [20, Theorems 3 and 3'] to (82) we get

$$\begin{aligned}-\int_{x_0}^{+\infty}\frac{(q_\infty^+(x))'}{2q^2(x)}\frac{y'^2(x)}{\rho(x)}dx &\geq K_1+\frac{1}{2}\sum_{k=\frac{i_1}{2}}^{\infty}\frac{-(q_\infty^+(\xi_{2k}))'}{h(\xi_{2k})}\frac{y^2(x_{2k+1})}{(\rho_0+\delta)\int_{x_{2k}}^{x_{2k+1}}\frac{q^2(x)}{h(x)}dx}\\ &\quad+\frac{1}{2}\sum_{k=\frac{i_1}{2}}^{\infty}\frac{-(q_\infty^+(\xi_{2k+1}))'}{h(\xi_{2k+1})}\frac{y^2(x_{2k+1})}{(\rho_0+\delta)\int_{x_{2k+1}}^{x_{2k+2}}\frac{q^2(x)}{h(x)}dx},\end{aligned} \tag{85}$$

since x_{2k+1} are the maxima of $y^2(x)$.

Since we assumed that $\rho_0<\infty$, from (19) it is clear that $\lim_{j\to+\infty}F(y(x_{2j+1}))=\rho_0$ and that $\rho_0-\delta<F(y(x_{2j+1}))<\rho_0-\delta$ for $j>\frac{i_1}{2}$. Since $f(y)$ is an odd function such that $f(y)>0$ for $y>0$, $F(y)$ must be even and strictly increasing for $y>0$. Therefore we can define a function $H(t)$ also increasing for $t>0$ such that $H(F(y))=y^2$. Accordingly $y^2(x_{2j+1})=H(F(y(x_{2j+1})))>H(\rho_0-\delta)>0$ for $j>\frac{i_1}{2}$. As a consequence from (80) and (85) one gets

$$\rho_0\geq\rho(x_0)\exp(G+K_1)\exp\left(\frac{H(\rho_0-\delta)}{2(\rho_0+\delta)}\sum_{i=i_1}^{\infty}\frac{\frac{-(q_\infty^+(\xi_i))'}{h(\xi_i)}}{\int_{x_i}^{x_{i+1}}\frac{q^2(x)}{h(x)}dx}\right). \tag{86}$$

Since from Theorem 5 the sequence of zeroes $\{x_i\}$ satisfies (68)–(70), from hypothesis (77) and (86) one gets that $\rho_0=\infty$ in contradiction with our assumption. Therefore $\rho_0=\infty$ and given that $\lim_{j\to+\infty}F(y(x_{2j+1}))=+\infty$, from (19) and the hypothesis one yields $\limsup_{x\to+\infty}y(x)=+\infty$ and $\liminf_{x\to+\infty}y(x)=-\infty$. $\square$

Corollary 4. *Under the same hypotheses of Theorem 6, if for each sequence $\{x_i,\ i \geq 1\}$ which satisfies (68)–(70) there exists a positive and piecewise continuous function $h(x)$ such that either*

$$\sum_{i=1}^{\infty} \frac{\int_{x_{2i}}^{x_{2i+1}} \min\left(\frac{-(q_\infty^+(s))'}{h(s)},\ x_{2i} \leq s \leq x\right) \frac{h(x)}{q^2(x)} dx}{\int_{x_{2i}}^{x_{2i+1}} \frac{h(x)}{q^2(x)} dx \int_{x_{2i}}^{x_{2i+1}} \frac{q^2(x)}{h(x)} dx} = +\infty, \tag{87}$$

or

$$\sum_{i=1}^{\infty} \frac{\int_{x_{2i+1}}^{x_{2i+2}} \min\left(\frac{-(q_\infty^+(s))'}{h(s)},\ x \leq s \leq x_{2i+2}\right) \frac{h(x)}{q^2(x)} dx}{\int_{x_{2i+1}}^{x_{2i+2}} \frac{h(x)}{q^2(x)} dx \int_{x_{2i+1}}^{x_{2i+2}} \frac{q^2(x)}{h(x)} dx} = +\infty, \tag{88}$$

then all solutions of (3) oscillatory on $]x_0, +\infty[$, if they exist, satisfy $\limsup_{x\to+\infty} y(x) = +\infty$ and $\liminf_{x\to+\infty} y(x) = -\infty$.

Proof. The proof is straightforward since (77) is the sum of (87) and (88). □

Corollary 5. *Under the same hypotheses of Theorem 6, if for each sequence $\{x_i,\ i \geq 1\}$ which satisfies (68)–(70), there exists a positive and piecewise continuous function $h(x)$ such that*

$$\sum_{i=1}^{\infty} \frac{\min\left\{\frac{-(q_\infty^+(s))'}{h(s)},\ x_i \leq s \leq x_{i+1}\right\}}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{h(x)} dx} = +\infty, \tag{89}$$

then all solutions of (3) oscillatory on $]x_0, +\infty[$, if they exist, satisfy $\limsup_{x\to+\infty} y(x) = +\infty$ and $\liminf_{x\to+\infty} y(x) = -\infty$.

Proof. The proof is also immediate since

$$\min\left\{\frac{-(q_\infty^+(s))'}{h(s)},\ x_{2i} \leq s \leq x_{2i+1}\right\} \leq \min\left\{\frac{-(q_\infty^+(s))'}{h(s)},\ x_{2i} \leq s \leq x\right\}, \quad x \in [x_{2i}, x_{2i+1}],$$

and

$$\min\left\{\frac{-(q_\infty^+(s))'}{h(s)},\ x_{2i-1} \leq s \leq x_{2i}\right\} \leq \min\left\{\frac{-(q_\infty^+(s))'}{h(s)},\ x \leq s \leq x_{2i}\right\}, \quad x \in [x_{2i-1}, x_{2i}]. \quad \square$$

Corollary 6. *Under the same hypotheses of Theorem 6, if for each sequence $\{x_i,\ i \geq 1\}$ which satisfies (68)–(70), there exists a positive and piecewise continuous function $h(x)$ such that*

$$\sum_{i=1}^{\infty} \frac{1}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{-(q_\infty^+(s))'} dx} = +\infty, \tag{90}$$

where the terms $\frac{1}{\int_{x_i}^{x_{i+1}} \frac{q^2(x)}{-(q_\infty^+(s))'} dx}$ are substituted by zero if $(q_\infty^+(x))'$ (ergo $q'(x)$) vanishes in $]x_i, x_{i+1}[$, then all solutions of (3) oscillatory on $]x_0, +\infty[$, if they exist, satisfy $\limsup_{x\to+\infty} y(x) = +\infty$ and $\liminf_{x\to+\infty} y(x) = -\infty$.

Proof. The proof is immediate from Corollary 5 by taking $h(x) = 1$ in those intervals $[x_i, x_{i+1}]$ having at least one point where $(q_\infty^+(x))'$ vanishes and $h(x) = -(q_\infty^+(x))'$ in the rest. □

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